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Note

Some new large sets of t-designs

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ABSTRACT

We investigate the existence of some large sets of size nine. We take advantage of the recursive and direct constructing method to show that in cases $LS[9](2, 4, v)$ the trivial necessary condition is also sufficient. In particular $LS[9](2, 4, 29)$ is constructed. This fills a gap.

1. Introduction

Let t, k, v and λ be positive integers such that $t \le k \le v$. Let X be a set of size v (or a v-set called point set) and $P_i(X)$, $0 < i \le t$, denote the set of all i-subsets of X. A t-(v, k, λ) design (or in short a t-design) is a pair (X, D) in which D is a collection of elements of $P_k(X)$ (called blocks) such that every t-subset of X appears in exactly λ blocks. A t-design is simple if no two blocks are identical. In this article, we consider only simple designs. The pair $(X, P_k(X))$ is called the complete design. A large set of t-(v, k, λ) designs, denoted by $LS[N](t, k, v)$, is a partition of the complete design in to N disjoint t-(v, k, λ) designs and hence $N = \binom{v-t}{k-t}/\lambda$. Consequently a necessary condition for the existence of an $LS[N](t, k, v)$ is that

$$N \mid \binom{v-i}{k-i}, \quad 0 \le i \le t. \tag{1.1}$$

This necessary condition (1.1) is not always sufficient, for in 1850, Cayley showed that it is possible to have two disjoint 2-(7, 3, 1) designs and no more [1]. So there are no $LS[5](2, 3, 7)$ and $LS[5](3, 4, 8)$.

The methods of constructing large sets can be divided in two categories: direct method and recursive method. The most common direct method for constructing t-designs and large sets is based on group actions. In this method one tries to find designs using a prescribed automorphism group. The block sets of such designs must be a union of some orbits of the prescribed group on all those subsets of the point set which are of the same size. This method was formulated by Kramer and Mesner as a matrix equation [3]. Here we exploit this method to construct some new large sets.

Let S_X be the permutation group on X. Let $\sigma \in S_X$ and $\beta \subseteq P_k(X)$. We denote by β^σ the image under σ of β. If $\beta^\sigma = \beta$, then σ is called an automorphism of β. If G is a subgroup of S_X such that $\beta^\sigma = \beta$ for every $\sigma \in G$, we say that β is G-invariant.

In order to define the same concept for large sets, consider the set $D = \{\beta_i\}_{i=1}^N$, such that each β_i is a t-(v, k, λ) design and $\beta_i \cap \beta_j = \emptyset$ for all $i \ne j$. A permutation $\sigma \in S_X$ is said to be an automorphisms of D if $D^\sigma = D$, that is, $\beta_i^\sigma \in D$, for each $\beta_i \in D$. The set of all automorphisms of D is, of course, a subgroup of S_X denoted by *Aut D*. If G is a subgroup of *Aut D*, we say that D is G-invariant.

Suppose that the block set β of a design is G-invariant for some abelian group G. Then one needs only list part of the blocks, called starter blocks, in order to obtain β. In other words starter blocks is a set of orbit representations in β under the action of G.

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Table 1
Starter blocks of 78 distinct 2-(29, 4, 3) designs.

D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}
0124	0125	0126	0127	0128	0129	012a	012b	012c	012d	012e	012f	0137
05aj	018j	018b	014c	0149	014a	014g	0158	015e	0169	014b	01aj	01cg
036g	02cf	039d	024f	02dg	024g	0249	024e	0247	024b	024c	0248	01cg
029k	049i	026i	039m	02ch	038j	038e	039g	039l	03dh	039e	036c	025f
048g	04ah	029h	03bh	037d	038f	03cj	038h	03ai	037m	038c	038g	026o
05bh	027n	03di	049j	04bl	04dj	03cj	038h	03ai	037m	038c	038g	03bk
018f	039h	05ck	05cl	05bk	05bm	05ci	05bj	05dk	06gk	06dk	05ah	05jd
D_{14}	D_{15}	D_{16}	D_{17}	D_{18}	D_{19}	D_{20}	D_{21}	D_{22}	D_{23}	D_{24}	D_{25}	D_{26}
012r	012q	012p	012o	012n	012m	012l	012k	012j	012i	012h	012g	01nr
05ak	01bm	01jm	018q	01lg	01kq	01eq	01mp	01gp	01lo	01jq	01bk	01am
036j	02gj	03jn	024i	02fi	024h	024o	024j	024q	024m	024l	024p	01ei
02bm	04fo	02dp	03am	02ej	03do	03io	03gn	03bn	03fj	03in	036n	02gq
048l	04gn	02em	03fl	03jp	03ho	03dk	03fo	03em	03ap	03ko	03go	027p
05hn	028o	03ej	04eo	04cm	04ek	04fn	04gm	04hn	05al	05dn	04fm	03cl
01fm	03fn	05em	05dm	05en	05cn	05gm	05fn	05el	06cl	06fm	05am	05fl
D_{27}	D_{28}	D_{29}	D_{30}	D_{31}	D_{32}	D_{33}	D_{34}	D_{35}	D_{36}	D_{37}	D_{38}	D_{39}
0138	0139	013a	013b	013c	013d	013e	013h	013i	013j	013k	013l	013m
014d	014e	014o	017g	014n	014m	014l	017a	014k	01ad	01be	014i	014j
015j	015c	01ce	015i	015d	016a	015b	018c	015m	015n	016l	016m	016d
028f	028m	026l	029c	027l	027f	029f	026b	027d	026e	026d	027h	026k
02af	02ci	03af	026n	02ag	027g	029l	02ah	028g	027k	025n	028d	028l
039k	038i	04ck	038m	039i	04al	037f	03bg	03ak	038n	048h	037k	03ch
04am	049g	05bi	04di	04bg	03ci	05ag	049n	049f	04cj	05cj	04aj	049m
D_{40}	D_{41}	D_{42}	D_{43}	D_{44}	D_{45}	D_{46}	D_{47}	D_{48}	D_{49}	D_{50}	D_{51}	D_{52}
01mr	01lr	01kr	01jr	01ir	01hr	01gr	01en	01cr	01br	01ar	019r	018r
01hq	01gq	016q	01en	017q	018q	019q	01kn	01aq	01hk	01gj	01cq	01bq
01bp	01ip	01gi	01cp	01hp	01gi	01jp	01im	018p	017p	019o	018o	01ho
02gn	029n	02ao	02jm	02ao	02go	02gm	02kp	02io	02hp	02ip	02eo	02bp
02gl	02dj	03hm	028p	02fl	02fo	02am	02el	02fn	02bo	028q	02in	02an
03cn	03eo	04dl	03ao	03en	04cn	03hp	03gl	03cm	039o	048k	03cp	03fk
04bn	04ho	05gn	04fk	04hm	03ek	05an	04ao	04io	04el	05fm	04en	04bo
D_{53}	D_{54}	D_{55}	D_{56}	D_{57}	D_{58}	D_{59}	D_{60}	D_{61}	D_{62}	D_{63}	D_{64}	D_{65}
013n	013o	013p	046	0157	015k	015l	016e	016g	016h	016k	018d	0135
015g	017c	01bh	018i	01jc	0168	01ac	018a	019b	0179	017j	019e	015h
01ai	01ae	019g	017l	01bg	017k	018e	017b	018h	01af	019d	01bd	017f
025l	02be	025b	025i	028h	02bf	025c	025h	026h	02ad	025d	0269	028e
025e	028i	026g	029e	025m	02ai	027i	02cg	02bg	0259	026j	028b	03aj
048f	03ah	038k	03bf	03ag	038h	036f	036b	036a	037j	026m	037h	03al
06cj	048d	04bj	04ak	048i	037c	048e	04bk	037o	04ci	037l	04af	049k
D_{66}	D_{67}	D_{68}	D_{69}	D_{70}	D_{71}	D_{72}	D_{73}	D_{74}	D_{75}	D_{76}	D_{77}	D_{78}
017r	016r	015r	01oq	01np	01ap	019p	01go	01eo	01do	01ao	01hm	01pr
01ep	01in	01dj	01cm	01il	01mo	01ik	01km	01jl	01ln	01bn	01gl	01dp
01ck	01gk	01el	019n	01ej	01dn	01gm	01jn	01dm	01fk	01hl	01hj	01fn
02aq	02hk	02kq	02dq	02en	02gk	02jq	02eq	02ep	02il	02iq	02mp	02hn
02hq	02dn	02fp	02hm	029q	02dl	02do	036o	02fk	02mq	02cq	02kn	03dm
048m	03fm	03co	03hl	03gm	036i	036k	04dm	036p	03dp	029p	03fp	03bm
06cm	048o	04em	04dn	048j	03kp	048n	02fj	038p	04fl	03bp	04in	04do

Let G be a subgroup of S_X and let $T_1, T_2, \dots, T_s$ and $K_1, K_2, \dots, K_r$ (for positive integers r and s) be the orbits of $P_t(X)$ and $P_k(X)$ under the action of G, respectively. The Kramer–Mesner matrix is the $s \times r$ matrix $A^v_{t,k}(G)$ whose (i,j)-th entry is

$$|\{k \in K_j; T \subseteq k\}|,$$

where T is any representative in T_i. The following theorem, due to Kramer and Mesner gives more details [3].

Theorem A. *There exists a G-invariant t-(v, k, λ) design (X, β) if and only if there exists a vector $u \in \{0, 1\}^r$ satisfying the equation*

$$A^v_{t,k}(G)u = \lambda J, \tag{1.2}$$

where J is the s-dimensional all one vector.

Notation. Let N, t and k be some positive integers such that $t \le k$. The set of all v for which an $LS[N](t, k, v)$ exists is denoted by $A[N](t, k)$. The set of all v which satisfy the necessary conditions (1.1) is denoted by $B[N](t, k)$.

Table 2

Starter blocks of a 2-(29, 4, 39) design (β_7)						Starter blocks of a 2-(29, 4, 39) design (β_8)					
0136	016*f*	019*h*	013*q*	026*f*	029*g*	01*or*	016*p*	01*cn*	01*ek*	014*r*	026*q*
013*f*	016*i*	019*i*	014*h*	026*j*	029*i*	01*fr*	01*jo*	01*an*	01*dk*	01*dq*	02*lp*
013*g*	016*j*	019*k*	015*f*	027*a*	029*j*	01*er*	01*io*	018*n*	01*fj*	01*fp*	02*jp*
0147	016*n*	01*ag*	024*a*	027*b*	02*ae*	01*nq*	01*fo*	01*em*	01*cj*	024*n*	02*gp*
0148	017*d*	01*ah*	024*d*	027*c*	02*aj*	01*mq*	01*co*	019*m*	019*j*	024*k*	02*cp*
014*f*	017*e*	01*bf*	025*a*	027*e*	02*ak*	01*fq*	01*bo*	01*fl*	01*fi*	02*lq*	02*lo*
014*p*	017*i*	01*ei*	025*g*	027*j*	02*bh*	015*q*	017*o*	01*dl*	01*di*	02*fq*	02*ko*
0159	017*k*	01*bl*	025*k*	027*m*	02*bi*	01*lp*	01*hn*	01*cl*	01*fh*	02*bq*	02*jo*
015*a*	017*m*	01*cf*	025*o*	028*c*	02*bj*	01*kp*	01*gn*	01*al*	01*eh*	027*q*	02*ho*
015*o*	018*g*	01*ch*	025*p*	028*j*	02*dh*	0258	03*lp*	03*ep*	03*hn*	03*im*	03*gk*
016*b*	018*l*	01*df*	026*a*	028*k*	02*nq*	036*m*	03*ip*	036*p*	03*bo*	03*el*	04*jo*
016*c*	019*f*	01*dj*	026*c*	029*d*	036*d*	036*l*	03*gp*	03*jo*	03*dn*	03*dl*	04*go*
03*bj*	03*cg*	049*e*	049*h*	049*l*	04*ch*	02*hl*	02*cl*	02*bl*	02*ek*	02*dk*	02*ck*
036*e*	037*b*	037*e*	037*g*	037*i*	037*n*	02*co*	029*o*	02*jn*	02*cn*	02*bn*	02*im*
038*d*	039*f*	038*l*	039*j*	03*ae*	03*bi*	02*fm*	02*dm*	02*cm*	04*co*	04*gl*	06*em*
06*dl*						02*ei*					

2. Large sets of size 9 with $t = 2$ and $k = 4$

In this section first we state the necessary condition (1.1), in terms of some congruency relations. Then we prove the existence of a family of large sets. Let p be a prime number and k_p be the smallest power of p such that $k < p^{k_p}$. The following theorem helps us to identify the existence of large sets in size 9 and $t = 2$, $k = 4$.

Theorem B ([5]). *Let p^α be a prime power. Then $v \in B[p^\alpha](t, k)$ if and only if one of the following conditions hold:*

i. $v \equiv t, \ldots, k-1 \pmod{p^{k_p+\alpha-1}}$,
ii. $v \equiv v_0 \pmod{p^{k_p+\alpha-1}}$, $k < v_0 < p^{k_p+\alpha-1}$ *and* $v_0 \in B[p^\alpha](t, k)$.

This theorem has, in particular, the following easy conclusion.

Lemma 1. *Let $v > 7$ then $v \in B[9](2, 4)$ if and only if* $v \equiv 2, 3 \pmod{27}$.

Proof. In this case $k_p = 2$ and one can easily see that for $6 < v_0 < 27$ we have

$$9 \nmid \binom{v-i}{4-i}, \quad i = 0, 1, 2.$$

Therefore by Theorem B, $v \in B[9](2, 4)$ if and only if $v \equiv 2, 3 \pmod{27}$. □

The following theorem due to Khosrovshahi et al. [2], presents a recursive method and provides an important mechanism for constructing families of large sets from small cases.

Theorem C ([2]). *If there exist $LS[N](t, i, v)$ for all $t + 1 \le i \le k$, then there exist $LS[N](t, i, l(v - t) + j)$ for all $l \ge 1$, $t + 1 \le i \le k$ and $t \le j < i$.*

In next theorem we take advantage of the direct and recursive method to obtain some new results on large sets of size 9.

Theorem 2. $B[9](2, 4) = A[9](2, 4)$.

Proof. Let $X = \{0, 1, \ldots, 9, a, b, \ldots, s\}$ and let the permutation group G of order 29 is generated by $(\,0\ 1\ \ldots\ 9\ a\ b\ \ldots\ s\,)$. By Theorem C and the existence of $LS[9](2, 3, 29)$ (see [4]), it suffices to establish the existence of $LS[9](2, 4, 29)$. In the status of $k = 4$, $t = 2$, $\lambda = 3$ and $v = 29$, the Kramer–Mesner matrix is of size 14×819 and we can find 78 disjoint binary solutions for the Eq. (1.2). So by Theorem A each of these solutions correspond to a 2-(29, 4, 3) design. In Table 1 we exhibit all 78 starter blocks of these designs in columns titled D_i, $1 \le i \le 78$. When G acts on all rows (blocks) of each column D_i, then a 2-(29, 4, 3) design is produced.

These 78 designs given in the Table 1 can be partitioned into six disjoint sets $\beta_1, \ldots, \beta_6$, each consisting of 13 blocks of a cyclic 2-(29, 4, 3) design. So each β_i, $i = 1, \ldots, 6$ is a cyclic 2-(29, 4, 39) design. By Theorem A one can easily show that when G acts on two families of starter blocks as are listed in the following table, two disjoint 2-(29, 4, 39) are produced. We call these two designs β_7 and β_8, respectively. The starter blocks of these two latter designs (β_7&β_8) are listed in columns of Table 2.

Since $(X, P_4(X))$ is a cyclic 2-(29, 4, 351) design, the set of blocks $(P_4(X) \setminus \bigcup_{i=1}^{8} \beta_i,)$ called β_9, is a cyclic 2-(29, 4, 39) design. Now the set $\{\beta_1, \ldots, \beta_9\}$ is an $LS[9](2, 4, 29)$. This completes the proof. □

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